Quantifying Superposition

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Measures are introduced to quantify the degree of superposition in mixed states with respect to orthogonal decompositions of the Hilbert space of a quantum system. These superposition measures can be regarded as analogues to entanglement measures, but can also be put in a more direct relation to the latter. By a second quantization of the system it is possible to induce superposition measures from entanglement measures. We consider the measures induced from relative entropy of entanglement and entanglement of formation. We furthermore introduce a class of measures with an operational interpretation in terms of interferometry. We consider the superposition measures under the action of subspace preserving and local subspace preserving channels. The theory is illustrated with models of an atom undergoing a relaxation process in a Mach-Zehnder interferometer.

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I. INTRODUCTION

In analogy with entanglement measures we develop means to quantify to what degree a mixed quantum state is in superposition with respect to given orthogonal subspaces of the Hilbert space of the system. We approach this question from two angles. In the first approach we pursue the analogies and relations to entanglement measures, while in the second we focus on the role of superposition in interferometry.

One specific example of a system in superposition is a single particle in superposition between two separated spatial regions. Such states can display nonlocality [1, 2, 3, 4, 5] and has been considered as a resource for teleportation [6] and quantum cryptography [7]. In these contexts, the single particle superposition is often not explicitly described as such, but is rather modeled using a second quantized description of the system, where the superposition between the two regions is viewed as an entangled state, formed using the vacuum and single particle states of the two regions. This example suggests that the relation between superposition and entanglement goes beyond mere analogy, since the former can be regarded as the latter, in the above sense. With this observation in mind we apply second quantizations in order to induce superposition measures from entanglement measures, and thus obtain, through the entanglement measures, a resource perspective on superposition. We investigate the superposition measures induced by relative entropy of entanglement [8, 9] and entanglement of formation [10].

Apart from the fundamental questions concerning superposition, a quantitative approach is relevant for the development of techniques to probe quantum processes using interferometry [11, 12, 13, 14, 15]. In this approach the processes are distinguished by how they affect the superposition of the probing particle in the interferometer. A systematic investigation of these phenomena thus calls for a quantitative understanding of mixed state superposition, which the present investigation may facilitate. To complement the class of induced superposition measures we consider measures that have a direct operational interpretation in terms of interference experiments.

Lately, a measure has been introduced [16] to quantify the interference of general quantum processes with respect to a given (computational) basis. The superposition measures considered here can be regarded as complementary to this approach, as we primarily focus on the properties of states, rather than processes.

The connection to interferometry suggests a close relation between superposition measures and subspace preserving (SP) and local subspace preserving (LSP) channels [11]. We analyze the change of superposition under the action of SP and LSP channels. The theory is illustrated with models of an atom that decays to its ground state while propagating through a Mach-Zehnder interferometer.

The structure of the paper is as follows. Section II introduces the concept of superposition measures by two specific examples; the relative entropy of superposition and the superposition of formation. In Sec. III we relate these measures to the relative entropy of entanglement and entanglement of formation. In Sec. IV we introduce the concept of induced superposition measures. In Sec. IV A we show that relative entropy of superposition is induced by relative entropy of entanglement, and in Sec. IVB we show that superposition of formation is induced by entanglement of formation. Section V introduces another class of superposition measures based on unitarily invariant operator norms, and in Sec. VA we consider measures based on Ky-Fan norms. In Sec. VB it is shown that all Ky-Fan norm based measures have an operational interpretation via interferometric measurements, and in Sec. VC we show that the Ky-Fan norm measures are bounded by predictability. Section VI illustrates the relation between superposition measures and

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SP and LSP channels. In Sec. VIA we consider the induced superposition measures under the action of these classes of channels, and in subsection VIB we similarly consider the superposition measures obtained from unitarily invariant norms. In Sec. VII the theory is illustrated with models of an atom undergoing relaxation in a Mach-Zehnder interferometer. The paper is ended with the Conclusions in Sec. VIII.

II. SUPERPOSITION MEASURES

To obtain the simplest possible illustration of the idea of superposition measures, consider a two-dimensional Hilbert space spanned by the two orthonormal vectors $|1\rangle$ and $|2\rangle$. Assume that we consider states ρ for which $\langle 1|\rho|1\rangle = 1/2$, i.e., the probability to find the system in either of the two states is 1/2. All such density operators can be written $\rho = (|1\rangle\langle 1| + |2\rangle\langle 2| + c|1\rangle\langle 2| + c^*|2\rangle\langle 1|)/2$ where the complex number c satisfies |c| < 1. Hence, c is the off-diagonal element when ρ is represented in the $\{|1\rangle, |2\rangle\}$ basis. In one extreme, |c|=1, the state is in a pure equal superposition between the two states. In the other extreme, c=0, the state is maximally mixed and there is no superposition. We can conclude that the off-diagonal element c describes the superposition, and it seems intuitively reasonable to take the quantity |c|as a measure of "how much" superposition there is between the two states $|0\rangle$ and $|1\rangle$. This observation generalizes to pairs of orthogonal subspaces, in the sense that the off-diagonal block carries the information concerning the superposition. In Sec. V we consider superposition measures based directly on this observation, while in the following we focus on the analogy with, and relation to, entanglement measures.

Given a finite-dimensional Hilbert space \mathcal{H} we consider a collection of K at least one-dimensional subspaces $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_K)$ such that $\bigoplus_{k=1}^K \mathcal{L}_k = \mathcal{H}$, i.e., we consider a collection of pairwise orthogonal subspaces that spans the entire Hilbert space. When we say that \mathcal{L} is a decomposition of \mathcal{H} we assume the above mentioned properties. We let P_k denote the projector corresponding to subspace \mathcal{L}_k , and define the channel

$$\Pi(\rho) = \sum_{k=1}^{K} P_k \rho P_k. \tag{1}$$

We write $\Pi_{\mathcal{L}}$ when we want to stress that Π is defined with respect to the decomposition \mathcal{L} . The effect of Π is to remove all the "off-diagonal blocks" $P_i \rho P_j$ from the density operator, leaving the "diagonal blocks" $P_i \rho P_i$ intact.

Superposition measures A are real-valued functions on the set of density operators on \mathcal{H} , and are defined with respect to some decomposition \mathcal{L} . To clarify with respect to which decomposition the measure is defined we write $A^{\mathcal{L}}$.

The following provides a list of properties that a superposition measure may satisfy. Note that we do not claim these properties to be "necessary conditions" for superposition measures, they are merely convenient and reasonable conditions that the measures we consider here do satisfy. For all density operators ρ on \mathcal{H} ,

C1: $A(\rho) \geq 0$,

C2: $A(\rho) = 0 \Leftrightarrow P_i \rho P_j = 0, \forall ij : i \neq j,$

C3: $A(U\rho U) = A(\rho)$, $U = \bigoplus_j U_j$, where U_j is unitary on \mathcal{L}_j ,

C4: A is convex,

Property C1 states that the function is non-negative on all states. Condition C2 says that $A(\rho)$ is zero if and only if ρ is block-diagonal with respect to the decomposition \mathcal{L} . Since such a block-diagonal state has no superposition between the subspaces in question, C2 thus seems a reasonable condition. The set of block diagonal states can be seen as the analogue to the separable states in the context of entanglement (further elaborated in Sec. IV). Property C3 states that A is invariant under unitary transformations on the subspaces in the decomposition. From an intuitive point of view this seems reasonable since such unitary operations do not change the "magnitude" (e.g., some norm) of the off-diagonal operators, and thus should not change the "amount" of superposition. This condition is analogous to the invariance of entanglement measures under local unitary operations. The last property, C4, states that the degree of superposition does not increase under convex combinations of states (as is the case for, e.g., |c|, as defined above).

A. Relative entropy of superposition

We define the relative entropy of superposition as

$$A_S(\rho) = S(\Pi(\rho)) - S(\rho), \tag{2}$$

where S denotes the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \ln \rho)$. Note that

$$A_S(\rho) = S(\rho||\Pi(\rho)), \tag{3}$$

where $S(\rho||\sigma) = \text{Tr}(\rho \ln \rho) - \text{Tr}(\rho \ln \sigma)$, is the relative entropy [17, 18, 19]. One can also show that

$$A_S(\rho) = \inf_{\Pi(\sigma) = \sigma} S(\rho||\sigma), \tag{4}$$

where the infimum is taken over all block diagonal density operators σ on \mathcal{H} . Hence, an equivalent definition of A_S is to minimize the relative entropy with respect to all block diagonal states. This may give some intuitive understanding of why A_S could be regarded as a superposition measure. Note also that Π is a mixing enhancing channel [19] that removes off diagonal blocks of the input

density operator ρ . Intuitively, the more "off diagonal" the density operator ρ is, with respect to the subspaces in \mathcal{L} , the larger is the difference between the two entropies $S(\Pi(\rho))$ and $S(\rho)$.

By using the properties of relative entropy [19] one can show that A_S satisfies conditions C1, C2, C3, and C4. Moreover, from the additivity and monotonicity of relative entropy [19] it follows that A_S is additive and monotone in the following sense.

• Additivity: Let $\mathcal{L}^{(1)}$ be a decomposition of $\mathcal{H}^{(1)}$ and let $\mathcal{L}^{(2)}$ be a decomposition of $\mathcal{H}^{(2)}$. If ρ is a density operator on $\mathcal{H}^{(1)}$ and σ a density operator on $\mathcal{H}^{(2)}$, then

$$A_S^{\mathcal{L}^{(1)} \otimes \mathcal{L}^{(2)}}(\rho \otimes \sigma) = A_S^{\mathcal{L}^{(1)}}(\rho) + A_S^{\mathcal{L}^{(2)}}(\sigma). \tag{5}$$

• Monotonicity: Let \mathcal{L} be a decomposition of \mathcal{H} . If ρ is a density operator on $\mathcal{H} \otimes \mathcal{H}_a$, then

$$A_S^{\mathcal{L}}(\operatorname{Tr}_a \rho) \le A_S^{\mathcal{L} \otimes \hat{1}_a}(\rho).$$
 (6)

B. Superposition of formation

In the following, when we say that $(\lambda_l, |\psi_l\rangle)_l$ is a decomposition of a density operator ρ we intend that all $|\psi_l\rangle$ are normalized, $\lambda_l \geq 0$, and that

$$\rho = \sum_{l} \lambda_{l} |\psi_{l}\rangle \langle \psi_{l}|. \tag{7}$$

We define the superposition of formation as

$$A_f(\rho) = \inf \sum_{l} \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|)), \tag{8}$$

where the infimum is taken with respect to all decompositions $(\lambda_l, |\psi_l\rangle)_l$ of the density operator ρ . In analogy with entanglement of formation one may note that A_f is obtained if we minimize the expected superposition (measured with A_S) needed to prepare ρ from pure state ensembles.

Note that the definition of A_f involves an infimum. Technically speaking there is a question whether there exists a decomposition $(\lambda_l, |\psi_l\rangle)_l$ such that $A_f(\rho) = \sum_l \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|))$. To avoid such technicalities we note that for every $\epsilon > 0$ there exists a decomposition $(\lambda_l, |\psi_l\rangle)_l$ of ρ such that

$$A_f(\rho) \le \sum_l \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|)) \le A_f(\rho) + \epsilon.$$
 (9)

We refer to such a decomposition as an ϵ -decomposition of ρ .

The relative entropy of superposition is bounded from above by the superposition of formation,

$$A_S(\rho) < A_f(\rho). \tag{10}$$

To show this we let $(\lambda_l, |\psi_l\rangle)_l$ be a ϵ -decomposition of ρ . The joint convexity of relative entropy [19], together with the fact that $\text{Tr}(\sigma \ln \Pi(\rho)) = \text{Tr}(\Pi(\sigma) \ln \Pi(\rho))$, can be used to show that

$$A_S(\rho) \le \sum_{l} \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|)) \le A_f(\rho) + \epsilon.$$
 (11)

If we let $\epsilon \to 0$, Eq. (10) follows.

Now we shall prove that the superposition of formation A_f satisfies conditions C1, C2, C3, and C4. Condition C1 follows directly from the construction. To prove condition C2, assume $A_f(\rho) = 0$. From Eq. (10) it follows that $A_S(\rho) = 0$, and we already know that this implies that $P_i \rho P_j = 0$, $i, j : i \neq j$. Conversely, if $P_i \rho P_j = 0$, $i, j : i \neq j$ it follows that we can find a decomposition of ρ where each vector is localized in one of the subspaces \mathcal{L}_k , which implies $A_f(\rho) = 0$. Concerning property C3, we note that if $U = \bigoplus_j U_j$ then $\Pi(U|\psi\rangle\langle\psi|U^{\dagger}) = U\Pi(|\psi\rangle\langle\psi|)U^{\dagger}$. Since von Neumann entropy is invariant under unitary transformations, condition C3 follows. To prove condition C4, let $\rho = \sum_{n=1}^N \mu_n \rho_n$ be a convex combination of density operators. For each n let $(\lambda_l^{(n)}, |\psi_l^{(n)}\rangle)_l$ be an ϵ -decomposition of ρ_n . If we use Eq. (9) it follows that

$$A_{f}(\rho) \leq \sum_{n} \sum_{l} \mu_{n} \lambda_{l}^{(n)} S(\Pi(|\psi_{l}^{(n)}\rangle\langle\psi_{l}^{(n)}|))$$

$$\leq \sum_{n} \mu_{n} A_{f}(\rho_{n}) + \epsilon.$$
 (12)

If we now let $\epsilon \to 0$ we find that A_f is convex. Thus, A_f satisfies condition C4.

• Subadditivity: Let $\mathcal{L}^{(1)}$ be a decomposition of $\mathcal{H}^{(1)}$ and let $\mathcal{L}^{(2)}$ be a decomposition of $\mathcal{H}^{(2)}$. If ρ is a density operator on $\mathcal{H}^{(1)}$ and σ a density operator on $\mathcal{H}^{(2)}$, then

$$A_f^{\mathcal{L}^{(1)} \otimes \mathcal{L}^{(2)}}(\rho \otimes \sigma) \le A_f^{\mathcal{L}^{(1)}}(\rho) + A_f^{\mathcal{L}^{(2)}}(\sigma). \tag{13}$$

• Monotonicity: Let ρ be a density operator on $\mathcal{H} \otimes \mathcal{H}_a$

$$A_f^{\mathcal{L}}(\operatorname{Tr}_a \rho) \le A_f^{\mathcal{L} \otimes \hat{1}_a}(\rho).$$
 (14)

Proof. To prove the subadditivity in Eq. (13) we let $(\lambda_l^{(1)}, |\psi_l^{(1)}\rangle)_l$ and $(\lambda_{l'}^{(2)}, |\psi_{l'}^{(2)}\rangle)_{l'}$ be ϵ -decompositions of ρ and σ , respectively. It follows that $(\lambda_l^{(1)}\lambda_{l'}^{(2)}, |\psi_l^{(1)}\rangle|\psi_{l'}^{(2)}\rangle)_{l,l'}$ is a decomposition of $\rho\otimes\sigma$ and hence

$$A_{f}^{\mathcal{L}^{(1)} \otimes \mathcal{L}^{(2)}}(\rho \otimes \sigma) \leq \sum_{l,l'} \lambda_{l}^{(1)} \lambda_{l'}^{(2)} S(\Pi^{(1)}(|\psi_{l'}^{(1)}\rangle \langle \psi_{l'}^{(1)}|)$$

$$\otimes \Pi^{(2)}(|\psi_{l}^{(2)}\rangle \langle \psi_{l}^{(2)}|))$$

$$\leq A_{f}^{\mathcal{L}^{(1)}}(\rho) + A_{f}^{\mathcal{L}^{(2)}}(\sigma) + 2\epsilon, \quad (15)$$

where we at the second inequality have used the additivity of the von Neumann entropy, followed by Eq. (9). If we let $\epsilon \to 0$ in Eq. (15) we obtain subadditivity.

Next, we turn to the monotonicity in Eq. (14). Let $(\lambda_l, |\psi_l\rangle)_l$ be an ϵ -decomposition of the density operator ρ on $\mathcal{H} \otimes \mathcal{H}_a$, then

$$A_f^{\mathcal{L}\otimes\hat{1}_a}(\rho) + \epsilon \geq \sum_{l} \lambda_l S([\Pi \otimes I_a](|\psi_l\rangle\langle\psi_l|))$$

$$= \sum_{l} \lambda_l H(\boldsymbol{p}^{(l)}), \qquad (16)$$

where H denotes the Shannon entropy, and where

$$\boldsymbol{p}^{(l)} = (\langle \psi_l | P_1 \otimes \hat{1}_a | \psi_l \rangle, \dots, \langle \psi_l | P_K \otimes \hat{1}_a | \psi_l \rangle). \tag{17}$$

Now, consider a Schmidt decomposition $|\psi_l\rangle = \sum_m \sqrt{r_m^{(l)}} |\chi_m^{(l)}\rangle |a_m^{(l)}\rangle$, where $\{|\chi_m^{(l)}\rangle\}_m$ is an orthonormal set in \mathcal{H} , and $\{|a_m^{(l)}\rangle\}_m$ is orthonormal in \mathcal{H}_a . We find that

$$\operatorname{Tr}(P_k \otimes \hat{1}_a |\psi_l\rangle\langle\psi_l|) = \sum_m r_m^{(l)} \operatorname{Tr}(P_k |\chi_m^{(l)}\rangle\langle\chi_m^{(l)}|). \quad (18)$$

Thus, if we let $\boldsymbol{p}_m^{(l)} = [\text{Tr}(P_k|\chi_m^{(l)}\rangle\langle\chi_m^{(l)}|)]_k$, it follows that $\boldsymbol{p}^{(l)} = \sum_m r_m^{(l)} \boldsymbol{p}_m^{(l)}$. Due to the concavity of the Shannon entropy it follows that $H(\boldsymbol{p}^{(l)}) \geq \sum_m r_m^{(l)} H(\boldsymbol{p}_m^{(l)})$. If we combine this with Eq. (16) we find

$$A_f^{\mathcal{L}\otimes\hat{1}_a}(\rho) + \epsilon \ge \sum_{l} \lambda_l \sum_{m} r_m^{(l)} H(\boldsymbol{p}_m^{(l)}). \tag{19}$$

Since $(\lambda_l, |\psi_l\rangle)_l$ is a decomposition of ρ it follows that $(\lambda_l r_m^{(l)}, |\chi_m^{(l)}\rangle)_{l,m}$ is a decomposition of $\text{Tr}_a \rho$. Moreover, $S(\Pi(|\chi_m^{(l)}\rangle\langle\chi_m^{(l)}|)) = H(\mathbf{p}_m^{(l)})$. With this in Eq. (19) we obtain

$$A_f^{\mathcal{L}\otimes\hat{1}_a}(\rho) + \epsilon \geq \sum_{l} \sum_{m} \lambda_l r_m^{(l)} S(\Pi(|\chi_m^{(l)}\rangle\langle\chi_m^{(l)}|))$$

$$\geq A_f^{\mathcal{L}}(\operatorname{Tr}_a \rho). \tag{20}$$

If we let $\epsilon \to 0$ we obtain the monotonicity. \square

III. RELATIONS BETWEEN SUPERPOSITION AND ENTANGLEMENT MEASURES

The results in the previous sections suggest an analogy between the relative entropy of superposition and relative entropy of entanglement, and similarly between superposition of formation and entanglement of formation. Here we show that there do exist relations between these measures on certain classes of states. The primary reason why we consider these types of states is that they arise when we in Sec. IV consider superposition measures induced by entanglement measures.

Consider a decomposition $\{\mathcal{L}_k^{(1)}\}_{k=1}^K$ of a finite-dimensional Hilbert space $\mathcal{H}^{(1)}$, with corresponding projectors $P_k^{(1)}$. Similarly, consider a decomposition $\{\mathcal{L}_k^{(2)}\}_{k=1}^K$ of a finite-dimensional Hilbert space $\mathcal{H}^{(2)}$, and

projectors $P_k^{(2)}$. We require the same number of subspaces in both collections.

On the total Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ we consider the following subspace

$$\overline{\mathcal{L}} = \bigoplus_{k=1}^{K} \overline{\mathcal{L}}_{k}, \quad \overline{\mathcal{L}}_{k} = \mathcal{L}_{k}^{(1)} \otimes \mathcal{L}_{k}^{(2)}, \tag{21}$$

i.e., the subspace $\overline{\mathcal{L}}$ is an orthogonal sum of product subspaces. Moreover we define the following decomposition of $\overline{\mathcal{L}}$,

$$\overline{\mathcal{L}} = \{\overline{\mathcal{L}}_k\}_{k=1}^K, \tag{22}$$

we denote the corresponding projectors as

$$\overline{P} = \sum_{k=1}^{K} \overline{P}_k, \quad \overline{P}_k = P_k^{(1)} \otimes P_k^{(2)}. \tag{23}$$

In this section we consider states ρ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ such that

$$\overline{P}\rho\overline{P} = \rho. \tag{24}$$

One may note that every bipartite pure state is of this type, due to the Schmidt decomposition. In this case the relevant subspaces are one-dimensional and correspond to the elements in the Schmidt decomposition.

A. Relative entropy of entanglement

The relative entropy of entanglement [9] for a bipartite state σ on a Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ is defined as $E_S(\sigma) = \inf_{\rho} S(\sigma||\rho)$, where the infimum is taken over all separable states $\rho = \sum_k \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)}$ with respect to two subsystems. If ρ is a separable state such that $E_S(\sigma) = S(\sigma||\rho)$, then we say that ρ is a minimizing separable state with respect to σ .

Lemma 1. Let ρ be a density operator on a Hilbert space \mathcal{H} and let $(\mathcal{L}_1, \mathcal{L}_2)$ be a decomposition of \mathcal{H} . Then ρ can be written

$$\rho = p_1 \sigma_1 + p_2 \sigma_2 + \sqrt{p_1 p_2} \sqrt{\sigma_1} D \sqrt{\sigma_2} + \sqrt{p_1 p_2} \sqrt{\sigma_2} D^{\dagger} \sqrt{\sigma_1},$$
(25)

where σ_1 and σ_2 are density operators such that $P_1\sigma_1P_1=\sigma_1,\ P_2\sigma_2P_2=\sigma_2,\ p_1,p_2\geq 0,\ p_0+p_1=1,$ and D satisfies $DD^\dagger\leq \hat{1}$.

Note that $DD^{\dagger} \leq \widehat{1}$ if and only if the largest singular value of D is less than or equal to 1. To prove Lemma 1 one can use Lemma 13 in Ref. [11], or a proof almost identical to the proof of Proposition 1 in Ref. [20]. Using Lemma 1 it is straightforward to prove the following: Lemma 2. Let $\{\mathcal{L}_k\}_{k=1}^K$ be a decomposition of a subspace \mathcal{L} of \mathcal{H} . Let P be the projector onto \mathcal{L} . If ρ is a density

operator on \mathcal{H} , such that $P\rho P = \rho$, then σ can be written

$$\rho = \sum_{k,k':k \neq k'} \sqrt{p_k p_{k'}} \sqrt{\sigma_k} D^{(kk')} \sqrt{\sigma_{k'}} + \sum_{k=1}^{k} p_k \sigma_k,$$
(26)

where σ_k are density operators such that $P_k \sigma_k P_k = \sigma_k$, $p_k \geq 0$, $\sum_k p_k = 1$, and $D^{(kk')}D^{(kk')} \leq \hat{1}$.

Note that this lemma only gives necessary conditions for ρ to be a density operator. It does not give sufficient conditions.

We define the function

$$f_{\rho^*}^{\sigma}(x,\rho) = S(\sigma||(1-x)\rho^* + x\rho),$$
 (27)

where σ , ρ , and ρ^* are density operators and $0 \le x \le 1$. In Ref. [9] it is shown that

$$\frac{\partial f_{\rho^*}^{\sigma}}{\partial x}(0,\rho) = 1 - \int_0^{\infty} \text{Tr}[Y_t(\rho^*,\sigma)\rho]dt, \qquad (28)$$

where

$$Y_t(\rho^*, \sigma) = (\rho^* + t\hat{1})^{-1}\sigma(\rho^* + t\hat{1})^{-1}.$$
 (29)

Moreover, the following result, which we rephrase as a lemma, is proved in Ref. [9].

Lemma 3. Let σ and ρ^* be density operators on \mathcal{H} , and let ρ^* be separable. If

$$\frac{\partial f_{\rho^*}^{\sigma}}{\partial x}(0,\rho) \ge 0,\tag{30}$$

for all pure product states ρ , then ρ^* has to be a minimizing separable state with respect to σ .

The fact that it is sufficient to satisfy Eq. (30) for pure product states only, follows from the linearity of the right hand side of Eq. (28) with respect to ρ . Note also that $\left|\frac{\partial}{\partial x} f_{\rho^*}^{\sigma}(0,\rho) - 1\right| \leq 1$ implies Eq. (30).

Lemma 4. If ρ^* is a minimizing separable state with respect to σ , then

$$\left| \frac{\partial f_{\rho^*}^{\sigma}}{\partial x}(0, \rho) - 1 \right| \le 1, \tag{31}$$

for all separable states ρ .

The absolute value in Eq. (31) implies that two inequalities have to be satisfied. One of the inequalities follows since ρ^* minimizes the convex function $S(\sigma|\rho)$ among all separable states. The other inequality follows from the fact that $\text{Tr}[Y_t(\rho^*,\sigma)\rho] \geq 0$.

We also need the following lemma:

Lemma 5. Let P_1 and P_2 be projectors onto subspaces of \mathcal{H}_1 and \mathcal{H}_2 , respectively. Suppose σ is a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $P_1 \otimes P_2 \sigma P_1 \otimes P_2 = \sigma$. Then every minimizing separable state ρ^* with respect to σ has to satisfy $P_1 \otimes P_2 \rho^* P_1 \otimes P_2 = \rho^*$.

Proof. Let $P_1^{\perp} = \hat{1}_1 - P_1$ and $P_2^{\perp} = \hat{1}_2 - P_2$. Now suppose ρ^* is a minimizing separable state with respect to σ , but such that $P_1 \otimes P_2 \rho^* P_1 \otimes P_2 \neq \rho^*$. It follows that $\text{Tr}(P_1 \otimes P_2 \rho^*) < 1$. Note also that without loss of generality we may assume $\text{Tr}(P_1 \otimes P_2 \rho^*) > 0$. Consider the channel

$$\Phi(\rho) = P_1 \otimes P_2 \rho P_1 \otimes P_2 + P_1^{\perp} \otimes P_2^{\perp} \rho P_1^{\perp} \otimes P_2^{\perp}
+ P_1 \otimes P_2^{\perp} \rho P_1 \otimes P_2^{\perp}
+ P_1^{\perp} \otimes P_2 \rho P_1^{\perp} \otimes P_2$$
(32)

and note that $\Phi(\sigma) = \sigma$. By using contractivity of relative entropy $S(\Phi(\sigma)||\Phi(\rho^*)) \leq S(\sigma||\rho^*)$ [21] together with $\Phi(\sigma) = \sigma$, one can show that $S(\sigma||\rho^*) > S(\sigma||\widetilde{\rho})$, where

$$\widetilde{\rho} = \frac{P_1 \otimes P_2 \rho^* P_1 \otimes P_2}{\text{Tr}(P_1 \otimes P_2 \rho^*)}.$$
(33)

Note that if since ρ^* is separable it follows that $\widetilde{\rho}$ is also separable. Hence we have found that ρ^* is not a minimizing separable state. This is a contradiction. Hence, $P_1 \otimes P_2 \rho^* P_1 \otimes P_2 = \rho^*$. This proves the lemma. \square Lemma 6. Let σ be a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\sigma = \overline{P} \sigma \overline{P}$. Then there exists a minimizing separable state ρ^* with respect to σ , such that

$$\rho^* = \sum_{k=1}^K \text{Tr}(\overline{P}_k \sigma) \rho_k^*, \tag{34}$$

where ρ_k^* is a minimizing separable state with respect to $\sigma_k = \overline{P}_k \sigma \overline{P}_k / \text{Tr}(\overline{P}_k \sigma)$, if $\text{Tr}(\overline{P}_k \sigma) \neq 0$. In the case $\text{Tr}(\overline{P}_k \sigma) = 0$ we let $\rho_k^* = 0$.

Proof. We assume that ρ^* is as in Eq. (34). To show that this is a minimizing separable state with respect to σ it is sufficient to prove that ρ^* satisfies Eq. (30) for all pure product states ρ . By the assumed form of ρ^* it follows that

$$\rho^* + t\hat{1} = \sum_{k=1}^{K} (p_k \rho_k^* + t\overline{P}_k) + \sum_{k,k':k \neq k'} tP_k^{(1)} \otimes P_{k'}^{(2)},$$
(35)

where $p_k = \text{Tr}(\overline{P}_k \sigma)$. From this it follows that

$$(\rho^* + t\hat{1})^{-1} = \sum_{k=1}^{K} (p_k \rho_k^* + t\overline{P}_k)^{\ominus} + \sum_{k,k': k \neq k'} t^{-1} P_k^{(1)} \otimes P_{k'}^{(2)}, \quad (36)$$

where X^{\ominus} denotes the Moore-Penrose (MP) pseudo inverse [22]. By Lemma 5 it follows that $\overline{P}_k \rho_k^* \overline{P}_k = \rho_k^*$. Hence, $\overline{P}_k (p_k \rho_k^* + t \overline{P}_k)^{\ominus} \overline{P}_k = (p_k \rho_k^* + t \overline{P}_k)^{\ominus}$.

By using Eq. (36), and
$$\overline{P}(P_k^{(1)} \otimes P_{k'}^{(2)}) = \delta_{kk'} P_k^{(1)} \otimes P_{k'}^{(2)}$$
,

together with Lemma 2, it follows that

$$Y_{t}(\rho^{*},\sigma) = \sum_{k,k'=1}^{K} (p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus} \sigma(p_{k'}\rho_{k'}^{*} + t\overline{P}_{k'})^{\ominus}$$

$$= \sum_{k=1}^{K} (p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus} p_{k}\sigma_{k}(p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus}$$

$$+ \sum_{k,k':k\neq k'} \sqrt{p_{k}p_{k'}} (p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus}$$

$$\times \sqrt{\sigma_{k}} D^{(kk')} \sqrt{\sigma_{k'}} (p_{k'}\rho_{k'}^{*} + t\overline{P}_{k'})^{\ominus}. \quad (37)$$

By combining Eqs. (28) and (37) we obtain

$$\left|1 - \frac{\partial f}{\partial x}(0, \rho)\right| \le \sum_{k=1}^{K} |A_k| + \sum_{kk': k \ne k'} |A_{kk'}|, \tag{38}$$

where

$$A_k = \int_0^\infty \text{Tr}[(p_k \rho_k^* + t\overline{P}_k)^{\ominus} p_k \sigma_k (p_k \rho_k^* + t\overline{P}_k)^{\ominus} \rho] dt, (39)$$

$$A_{kk'} = \sqrt{p_k p_{k'}} \int_0^\infty \text{Tr}[(p_k \rho_k^* + t \overline{P}_k)^{\ominus} \sqrt{\sigma_k} \times D^{(kk')} \sqrt{\sigma_{k'}} (p_{k'} \rho_{k'}^* + t \overline{P}_{k'})^{\ominus} \rho] dt. \quad (40)$$

By the change of variables $t = p_k s$ Eq. (39) can be rewritten as

$$A_{k} = \operatorname{Tr}(\overline{P}_{k}\rho) \int_{0}^{\infty} \operatorname{Tr}[(\rho_{k}^{*} + s\overline{P}_{k})^{\ominus} \sigma_{k} (\rho_{k}^{*} + s\overline{P}_{k})^{\ominus} \overline{\rho}_{k}] ds,$$
(41)

where

$$\overline{\rho}_k = \begin{cases} \overline{P}_k \rho \overline{P}_k / \text{Tr}(\overline{P}_k \rho), & \text{Tr}(\overline{P}_k \rho) \neq 0, \\ \hat{0}, & \text{Tr}(\overline{P}_k \rho) = 0. \end{cases}$$
(42)

Note that if ρ is separable, then $\overline{\rho}_k$ is separable. From Eqs. (41) and (28) it follows that

$$|A_k| = \text{Tr}(\overline{P}_k \rho) \left| 1 - \frac{\partial}{\partial x} f_{\rho_k^*}^{\sigma_k}(0, \overline{\rho}_k) \right| \le \text{Tr}(\overline{P}_k \rho), \quad (43)$$

where the inequality follows from Lemma 4 and the fact that $\overline{\rho}_k$ is separable.

We now turn to the $A_{kk'}$ term in Eq. (40). We make a singular value decomposition [22] of the operator $D^{(kk')}$, i.e., there exist orthonormal bases $\{|\alpha:n^{(kk')}\rangle\}_n$ and $\{|\beta:n^{(kk')}\rangle\}_n$ such that

$$D^{(kk')} = \sum_{n} r_n^{(kk')} |\alpha : n^{(kk')}\rangle \langle \beta : n^{(kk')}|, \qquad (44)$$

where $r_n^{(kk')} \geq 0$. (For convenience we allow singular values with value zero.) Since $D^{(kk')}D^{(kk')\dagger} \leq \hat{1}_{kk'}$ it follows that $r_n^{(kk')} \leq 1$. We assume ρ to be a pure and

separable state $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle$. We use this to find

$$|A_{kk'}| \le \sum_{n} \left| \int_{0}^{\infty} R_n^{(kk')}(t) Q_{k:n}^{(kk')^*}(t) dt \right|,$$
 (45)

where

$$Q_n^{(kk')}(t) = \sqrt{p_k} \langle \alpha : n^{(kk')} | \sqrt{\sigma_k} (p_k \rho_k^* + t \overline{P}_k)^{\ominus} | \psi \rangle.$$

$$R_n^{(kk')}(t) = \sqrt{p_{k'}} \langle \beta : n^{(kk')} | \sqrt{\sigma_{k'}} (p_{k'} \rho_{k'}^* + t \overline{P}_{k'})^{\ominus} | \psi \rangle.$$
(46)

We note that $Q_n^{(kk')}$ and $R_n^{(kk')}$ are $L^2(0,\infty)$ functions (which follows from Eq. (48)). Thus the Cauchy-Schwarz inequality is applicable to the right hand side of Eq. (45). On the result of this first application of the Cauchy-Schwarz inequality, $\sum_n [\int_0^\infty |R_n^{(kk')}(t)|^2 dt]^{1/2} [\int_0^\infty |Q_n^{(kk')}(t')|^2 dt']^{1/2}$, we again apply the Cauchy-Schwarz inequality, but this time on this expression regarded as an inner product of two finite vectors. This results in the upper bound

$$|A_{kk'}| \leq \sqrt{\sum_{n} \int_{0}^{\infty} |R_{n}^{(kk')}(t)|^{2} dt} \times \sqrt{\sum_{n'} \int_{0}^{\infty} |Q_{n'}^{(kk')}(t')|^{2} dt'}.$$
 (47)

Note that $P_{\alpha}^{(kk')} = \sum_{n} |\alpha:n^{(kk')}\rangle\langle\alpha:n^{(kk')}|$ is a projection operator. Now we use Eq. (46) and $P_{\alpha}^{(kk')} \leq \hat{1}$, to find

$$\sum_{n} \int_{0}^{\infty} |Q_{n}^{(kk')}(t)|^{2} dt = p_{k} \int_{0}^{\infty} \operatorname{Tr}[(p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus} \times \sigma_{k}(p_{k}\rho_{k}^{*} + t\overline{P}_{k})^{\ominus} \rho] dt.$$

On the right hand side of the above equation we recognize A_k , and thus by Eq. (43) it follows that

$$\sum_{n} \int_{0}^{\infty} |Q_{n}^{(kk')}(t)|^{2} dt \le \operatorname{Tr}(\overline{P}_{k}\rho). \tag{48}$$

By an analogous reasoning we find that $\sum_n \int_0^\infty |R_n^{(kk')}(t)|^2 dt \leq \operatorname{Tr}(\overline{P}_{k'}\rho)$. We combine the above equations with Eqs. (38), (43), and (47) we obtain

$$|1 - \frac{\partial f}{\partial x}(0, \rho)| \le \left(\sum_{k=1}^{K} \sqrt{\text{Tr}(\overline{P}_k \rho)}\right)^2.$$
 (49)

Now we again make use of the assumption that ρ is a (pure) product state $\rho = \rho_1 \otimes \rho_2$, and that $\overline{P}_k = P_k^{(1)} \otimes P_k^{(2)}$, to show that

$$\left| 1 - \frac{\partial f}{\partial x}(0, \rho) \right| = \left(\sum_{k=1}^{K} \sqrt{\operatorname{Tr}(P_k^{(1)} \rho_1)} \sqrt{\operatorname{Tr}(P_k^{(2)} \rho_2)} \right)^2 \le 1, \tag{50}$$

where we in the last inequality have used the Cauchy-Schwartz inequality. According to Lemma 3 it follows that ρ^* is a minimizing separable state with respect to σ . This proves the lemma. \square

<u>Proposition</u> 1. If σ is a density operator such that $\overline{P}\sigma\overline{P}=\sigma$, then

$$E_{S}(\sigma) = \sum_{k} \operatorname{Tr}(\overline{P}_{k}\sigma) E_{S}\left(\frac{\overline{P}_{k}\sigma\overline{P}_{k}}{\operatorname{Tr}(\overline{P}_{k}\sigma)}\right) + A_{S}^{\overline{L}}(\sigma).$$
(51)

In case $\text{Tr}(\overline{P}_k\sigma) = 0$, the corresponding term in the above sum is zero.

Proof. Since $\overline{P}\sigma\overline{P}=\sigma$ it follows by Lemma 6 that there exists a minimizing separable state ρ^* as in Eq. 34, and hence $E_S(\sigma)=S(\sigma||\rho^*)$. Now we can use the properties of ρ^* as defined in Lemma 6, noting that $\overline{P}_k\rho_k^*\overline{P}_k=\rho_k^*$, to calculate $S(\sigma||\rho^*)$ to be the right hand side of Eq. (51). This proves the proposition. \square

With respect to the here considered class of states Proposition 1 simplifies the calculation of the relative entropy of entanglement by breaking down the problem to the smaller subspaces $\overline{\mathcal{L}}_k$. If the marginal states ρ_k are separable, Proposition 1 thus provides a closed expression for the relative entropy of entanglement of the total state. The following gives a noteworthy special case.

Proposition 2. If $\overline{P}\sigma\overline{P} = \sigma$, and if at least one of the subspaces in each pair $(\mathcal{L}_k^{(1)}, \mathcal{L}_k^{(2)})$ is one-dimensional, then

$$E_S(\sigma) = A_S^{\overline{\mathcal{L}}}(\sigma). \tag{52}$$

This proposition follows directly from Proposition 1 since one of the subspaces $\mathcal{L}_k^{(1)}$ and $\mathcal{L}_k^{(2)}$ is one-dimensional, and hence $\overline{P}_k \sigma \overline{P}_k / \text{Tr}(\overline{P}_k \sigma)$ necessarily is a product state, and thus has zero entanglement.

B. Entanglement of formation

The entanglement of formation [10] of a density operator ρ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ is defined as

$$E_f(\rho) = \inf \mathcal{S}[(\lambda_l, |\psi_l\rangle)_l],$$
 (53)

where the infimum is taken over all decompositions $(\lambda_l, |\psi_l\rangle)_l$ of ρ , and where

$$S[(\lambda_l, |\psi_l\rangle)_l] \equiv \sum_l \lambda_l S(\text{Tr}_2(|\psi_l\rangle\langle\psi_l|)).$$
 (54)

Proposition 3. If ρ is a density operator such that $\overline{P}\rho\overline{P}=\rho$ then

$$E_{f}(\rho) \geq \sum_{k} \operatorname{Tr}(\overline{P}_{k}\rho) E_{f}\left(\frac{\overline{P}_{k}\rho \overline{P}_{k}}{\operatorname{Tr}(\overline{P}_{k}\rho)}\right) + A_{f}^{\overline{\mathcal{L}}}(\rho).$$

$$(55)$$

In case $\text{Tr}(\overline{P}_k \rho) = 0$, the corresponding term in the above sum is zero.

Proof. Consider an arbitrary decomposition $(\lambda_l, |\psi_l\rangle)_l$ of ρ . Let $p_k = \text{Tr}(\overline{P}_k \rho)$, and let

$$\sigma_k = \begin{cases} \overline{P}_k \rho \overline{P}_k / p_k, & p_k \neq 0, \\ \hat{0}, & p_k = 0. \end{cases}$$
 (56)

Similarly, let $p_l^{(k)} = \langle \psi_l | \overline{P}_k | \psi_l \rangle$, and

$$|\psi_l^{(k)}\rangle = \begin{cases} \overline{P}_k |\psi_l\rangle / \sqrt{p_l^{(k)}}, & p_l^{(k)} \neq 0, \\ 0, & p_l^{(k)} = 0. \end{cases}$$
 (57)

Now we note that $p_k = \sum_l \lambda_l p_l^{(k)}$. Hence, if we define $r_l^{(k)} = \lambda_l p_l^{(k)}/p_k$, we find that $\sum_l r_l^{(k)} = 1$. Moreover, we find that $(r_l^{(k)}, |\psi_l^{(k)}\rangle)_l$ is a decomposition of σ_k . Note that since $\overline{P}\rho\overline{P} = \rho$ it follows that $\overline{P}|\psi_l\rangle\langle\psi_l|\overline{P} = |\psi_l\rangle\langle\psi_l|$ for all l. The structure of the subspace $\overline{\mathcal{L}}$ implies that $\mathrm{Tr}_2(|\psi_l\rangle\langle\psi_l|) = \sum_k p_l^{(k)}\mathrm{Tr}_2(|\psi_l^{(k)}\rangle\langle\psi_l^{(k)}|)$. One can furthermore show that $S(\Pi(|\psi_l\rangle\langle\psi_l|)) = -\sum_k p_l^{(k)}\ln p_l^{(k)}$. By combining these facts it is possible to show that

$$S[(\lambda_l, |\psi_l\rangle)_l] = \sum_k p_k S[(r_l^{(k)}, |\psi_l^{(k)}\rangle)_l] + \sum_l \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|)).$$
 (58)

In order to calculate E_f we have to find the infimum of Eq. (58) over all decompositions $(\lambda_l, |\psi_l\rangle)_l$ of ρ . Note that all p_k are fixed by the choice of ρ .

$$E_{f}(\rho) = \inf_{(\lambda_{l}, |\psi_{l}\rangle)_{l}} \left[\sum_{k} p_{k} \mathcal{S}[(r_{l}^{(k)}, |\psi_{l}^{(k)}\rangle)_{l}] + \sum_{l} \lambda_{l} S(\Pi(|\psi_{l}\rangle\langle\psi_{l}|)) \right]$$

$$\geq \sum_{k} p_{k} \inf_{(r_{l}^{(k)}, |\psi_{l}^{(k)}\rangle)_{l}} \mathcal{S}[(r_{l}^{(k)}, |\psi_{l}^{(k)}\rangle)_{l}]$$

$$+ \inf_{(\lambda_{l}, |\psi_{l}\rangle)_{l}} \sum_{l} \lambda_{l} S(\Pi(|\psi_{l}\rangle\langle\psi_{l}|))$$

$$= \sum_{k} p_{k} E_{f}(\sigma_{k}) + A_{f}(\rho), \qquad (59)$$

where the infima are taken with respect to the decompositions $(\lambda_l, |\psi_l\rangle)_l$ of ρ , and $(r_l^{(k)}, |\psi_l^{(k)}\rangle)_l$ of σ_k . This proves the proposition. \square

Note that it is not clear whether the inequality in Eq. (55) can be replaced with an equality, or if there exist states where the inequality is strict. One can show, however, that when ρ is pure, then equality holds in Proposition 3. Another special case when equality holds is given by the following proposition.

Proposition 4. If ρ is a density operator such that $\overline{P}\rho\overline{P} = \rho$ and if at least one of the subspaces in each pair $(\mathcal{L}_k^{(1)}, \mathcal{L}_k^{(2)})$ is one-dimensional, then

$$E_f(\rho) = A_f^{\overline{\mathcal{L}}}(\rho). \tag{60}$$

Proof. If we neglect all terms but $A_f(\rho)$ on the right hand side of Eq. (55) we find that $E_f(\rho) \geq A_f(\rho)$. Let $(\lambda_l, |\psi_l\rangle)_l$ be an ϵ -decomposition of ρ with respect to A_f . We know that $(\lambda_l, |\psi_l\rangle)_l$ satisfies Eq. (58), where $|\psi_l^{(k)}\rangle = \overline{P}_k |\psi_l\rangle/||\overline{P}_k |\psi_l\rangle||$. Note that since at least one of the subspaces in each pair $(\mathcal{L}_k^{(1)}, \mathcal{L}_k^{(2)})$ is one-dimensional, it follows that each $|\psi_l^{(k)}\rangle$ is a pure product state. Hence, $S(\operatorname{Tr}_2(|\psi_l^{(k)}\rangle\langle\psi_l^{(k)}|)) = 0$, which we can insert in Eq. (58) to find that $E_f(\rho) \leq S[(\lambda_l, |\psi_l\rangle)_l] \leq \sum_l \lambda_l S(\Pi(|\psi_l\rangle\langle\psi_l|)) \leq A_f(\rho) + \epsilon$. If we now let $\epsilon \to 0$ we find $E_f(\rho) \leq A_f(\rho)$. If we combine this with our earlier finding that $E_f(\rho) \geq A_f(\rho)$, it follows that $E_f(\rho) = A_f(\rho)$ and the proposition is proved. \square

IV. INDUCED SUPERPOSITION MEASURES

In the second quantized description of a quantum system we describe the occupation states of an orthonormal basis of the original "first quantized" Hilbert space \mathcal{H} of the system, i.e., to each element in the basis we associate a Hilbert space that describes the possible pure occupation states. The tensor product of these spaces is the second quantized space $\mathcal{F}^{(x)}(\mathcal{H})$, where the x denotes the type of second quantization (bosonic, fermionic).

In the general case we consider a Hilbert space \mathcal{H} and a decomposition \mathcal{L} with K elements. The corresponding occupation number representation can be regarded as a K-fold tensor product of second quantizations of the subspaces, i.e., $\mathcal{F} = \bigotimes_{k=1}^K F^{(x)}(\mathcal{L}_k)$. Since we only consider single particle states it follows that the type of second quantization is irrelevant. Moreover, it follows that it is sufficient to restrict the analysis to the following subspace of \mathcal{F} . We extend each space \mathcal{L}_k with a vacuum state $\widetilde{\mathcal{L}}_k = \mathcal{L}_k \oplus \operatorname{Sp}\{|0\rangle\}$ and construct the space $\widetilde{\mathcal{H}} = \bigotimes_{k=1}^K \widetilde{\mathcal{L}}_k \subseteq \mathcal{F}(\mathcal{H})$. Since the single particle states are all elements of this subspace it suffices if we restrict the analysis to $\widetilde{\mathcal{H}}$.

Let $\{|k:l\rangle\}_{l=1}^{N_k}$ be an arbitrary but fixed orthonormal basis of subspace \mathcal{L}_k . To describe the transition from \mathcal{H} to \widetilde{H} it is convenient to use the following operator

$$M = \sum_{k=1}^{K} \sum_{l=1}^{N_k} |\widetilde{0}\rangle^{\otimes(k-1)} |\widetilde{1}_{k:l}\rangle |\widetilde{0}\rangle^{\otimes(K-k)} \langle k:l|, \qquad (61)$$

where $|\widetilde{0}\rangle$ denotes the vacuum state, and $|\widetilde{1}_{k:l}\rangle$ a single particle occupation of mode l in subsystem k. If ρ is a density operator on the first quantized space, then $M\rho M^{\dagger}$ is a density operator on the second quantized space. One can note that M is a linear isometry that maps \mathcal{H} to the single-particle subspace $\widetilde{\mathcal{H}}_{\text{single}}$ of $\widetilde{\mathcal{H}}$, i.e., $M^{\dagger}M=\hat{1}_{\mathcal{H}}$ and $MM^{\dagger}=P_{\text{single}}$, where P_{single} denotes the projector onto $\widetilde{\mathcal{H}}_{\text{single}}$.

As an illustration, consider a single particle that can be in two orthogonal orbitals, represented by the orthonormal vectors $|1\rangle$ and $|2\rangle$. The mapping in Eq. (61) takes

the form $M=|\widetilde{1}_1\rangle|\widetilde{0}_2\rangle\langle 1|+|\widetilde{0}_1\rangle|\widetilde{1}_2\rangle\langle 2|$. Hence, $|1\rangle$ and $|2\rangle$ are mapped to the product states $|\widetilde{1}_1\rangle|\widetilde{0}_2\rangle$ and $|\widetilde{0}_1\rangle|\widetilde{1}_2\rangle$. A pure superposition $\alpha|1\rangle+\beta|2\rangle$ is mapped to the pure entangled state $\alpha|\widetilde{1}\rangle|\widetilde{0}\rangle+\beta|\widetilde{0}\rangle|\widetilde{1}\rangle$. Finally, a diagonal state $\lambda_1|1\rangle\langle 1|+\lambda_2|2\rangle\langle 2|$ is mapped to the separable state $\lambda_1|\widetilde{1}\rangle\langle \widetilde{1}|\otimes|\widetilde{0}\rangle\langle \widetilde{0}|+\lambda_2|\widetilde{0}\rangle\langle \widetilde{0}|\otimes|\widetilde{1}\rangle\langle \widetilde{1}|$. This correspondence can be shown to hold also in the more general case. Given a density operator ρ on $\mathcal H$ and a K-element decomposition $\mathcal L$ of $\mathcal H$, the following holds

- ρ is localized in one subspace, i.e., $P_j \rho P_j = \rho$ for some j, if and only if $M \rho M^{\dagger}$ can be written as a product state $\widetilde{\rho}_1 \otimes \ldots \otimes \widetilde{\rho}_K$ with respect to all K subsystems.
- ρ is block diagonal with respect to \mathcal{L} , i.e., $\Pi_{\mathcal{L}}(\rho) = \rho$, if and only if $M\rho M^{\dagger}$ is completely disentangled with respect to all K subsystems, i.e., that it can be written as a convex combination of K-fold tensor products of density operators.

We are now ready to define induced superposition measures. Let M be defined by Eq. (61) with respect to a K-fold orthogonal decomposition of a Hilbert space \mathcal{H} , and let E be an K-partite entanglement measure. Define

$$A(\rho) = E(M\rho M^{\dagger}), \tag{62}$$

for all density operators ρ on \mathcal{H} . We say that A is the superposition measure induced by the entanglement measure E.

At first sight this definition may seem problematic since the operator M is not unique. M depends on arbitrary choices of orthonormal bases in the subspaces of the first quantized spaces. Thus we may construct $\widetilde{M} = MU$ where $U = \bigoplus_{k=1}^K U_k$ that would give rise to a new superposition measure. However, if the entanglement measure is invariant under local unitary transformations (often regarded as a requirement for a "good" entanglement measure [8]), it can be shown that the induced superposition measure is invariant under the various choices of M.

Note that if A is induced by an entanglement measure E, then A satisfies C1 if $E(\rho) \geq$ for all density operators ρ . Moreover, A satisfies C2 if E is such that $E(M\rho M^{\dagger}) = 0$ if and only if $M\rho M^{\dagger}$ is a completely disentangled state. Finally, A satisfies C3 if E invariant under local unitary operations, and A satisfies C4 if E is convex on the single particle subspace.

A. Measure induced by relative entropy of entanglement

For two complementary subspaces $\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{H}$ the mapping M maps \mathcal{H} to the (single particle) subspace $\overline{\mathcal{L}} = \mathcal{L}_1 \otimes \operatorname{Sp}\{|\widetilde{0}_2\rangle\} \oplus \operatorname{Sp}\{|\widetilde{0}_1\rangle\} \otimes \mathcal{L}_2$ of $\widetilde{H} = \widetilde{\mathcal{L}}_1 \otimes \widetilde{\mathcal{L}}_2$. One can see that $\widetilde{\mathcal{L}}$ is a special case of the class of subspaces considered in Sec. III. We let \overline{P} denote the projector onto the subspace $\overline{\mathcal{L}}$.

Proposition 5. With respect to a decomposition $(\mathcal{L}_1, \mathcal{L}_2)$ of \mathcal{H} the superposition measure induced by the bipartite relative entropy of entanglement E_S is the relative entropy of superposition A_S .

Proof. First, we note that the elements in the decomposition $\overline{\mathcal{L}} = (\mathcal{L}_1 \otimes \operatorname{Sp}\{|\widetilde{0}_2\rangle\}, \operatorname{Sp}\{|\widetilde{0}_1\rangle\} \otimes \mathcal{L}_2)$ are such that at least one of the subspaces in each product subspace is one-dimensional. Next, we note that $\overline{P}M\rho M^{\dagger}\overline{P} = M\rho M^{\dagger}$. Thus, Proposition 2 is applicable, and we can conclude that $E_S(M\rho M^{\dagger}) = A_S^{\overline{L}}(M\rho M^{\dagger})$. It remains to show that $A_S^{\overline{L}}(M\rho M^{\dagger}) = A_S^{\mathcal{L}}(\rho)$. To prove this we first note that

$$S(M\rho M^{\dagger}) = S(\rho), \tag{63}$$

which holds since M is a linear isometry and thus preserves the eigenvalues of ρ . Moreover, one can show that $\Pi_{\overline{\mathcal{L}}}(M\rho M^{\dagger}) = M\Pi_{\mathcal{L}}(\rho)M^{\dagger}$. If we combine this with Eq. (63) we find

$$S(\Pi_{\overline{\mathcal{L}}}(M\rho M^{\dagger})) = S(\Pi_{\mathcal{L}}(\rho)). \tag{64}$$

If we combine the definition of A_S with Eqs. (63) and (64) we find that $A_{\overline{S}}(M\rho M^{\dagger}) = A_S^{\mathcal{L}}(\rho)$. \square

B. Measure induced by entanglement of formation

Proposition 6. With respect to a decomposition $(\mathcal{L}_1, \mathcal{L}_2)$ of \mathcal{H} the superposition measure induced by the bipartite entanglement of formation E_f is the superposition of formation A_f .

Proof. The proof is analogous with the proof of the previous proposition, upon the use of Proposition 4, and up to the point where we have to prove that $A_f^{\overline{L}}(M\rho M^{\dagger}) = A_f^{\underline{L}}(\rho)$. If $(\lambda_l, |\psi_l\rangle)_l$ is a decomposition of ρ , then $(\lambda_l, |\widetilde{\psi}_l\rangle)_l$, with $|\widetilde{\psi}_l\rangle = M|\psi_l\rangle$, is a decomposition of $M\rho M^{\dagger}$. Vice versa, given a decomposition $(\lambda_l, |\widetilde{\psi}_l\rangle)_l$ of $M\rho M^{\dagger}$, it follows that $(\lambda_l, |\widetilde{\psi}_l\rangle)_l$ with $|\psi_l\rangle = M^{\dagger}|\widetilde{\psi}_l\rangle$ is a decomposition of ρ . Moreover, due to Eq. (64) we find that

$$\sum_{l} \lambda_{l} S(\Pi_{\overline{\mathcal{L}}}(|\widetilde{\psi}_{l}\rangle\langle\widetilde{\psi}_{l}|)) = \sum_{l} \lambda_{l} S(\Pi_{\mathcal{L}}(|\psi_{l}\rangle\langle\psi_{l}|)). \quad (65)$$

It follows that $A_f^{\overline{\mathcal{L}}}(M\rho M^{\dagger}) = A_f^{\mathcal{L}}(\rho)$. \square

The induced superposition measure inherit many of the properties of the corresponding entanglement measure. This may at first sight seem as an alternative technique to prove the properties of A_S and A_f proved in Sec. II. However, since we here only consider bipartite entanglement measures, it means that the induced superposition measures are defined only with respect to pairs of orthogonal subspaces. Thus, the derivations in Sec. II are more general.

We furthermore note that we can induce superposition measures form any other entanglement measure. One can for example show that the superposition measure induced from entanglement cost [23] satisfies the condition C1, C2, C3, and C4.

V. SUPERPOSITION MEASURES FROM UNITARILY INVARIANT NORMS

Given a decomposition $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ of a Hilbert space \mathcal{H} , a density operator ρ on \mathcal{H} can be decomposed into two diagonal operators $P_1\rho P_1$ and $P_2\rho P_2$, and two offdiagonal operators $P_1 \rho P_2$ and $P_2 \rho P_1$. In some sense the off-diagonal operators describes the superposition between the two subspaces. It thus seems reasonable to quantify the "amount" of superposition between the two subspaces by some measure of the magnitude of the offdiagonal operator. We consider superposition measures $A_u(\rho) = ||P_1 \rho P_2||$, where $||\cdot||$ is a norm on the space $\mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$ of linear operators from \mathcal{L}_1 to \mathcal{L}_2 . In order to make the superposition measure invariant under unitary transformations within the subspaces we assume that the norm $||\cdot||$ is unitarily invariant [24], i.e., such that $||U_2CU_1|| = ||C||$, for all $C \in \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$, and all unitary operators U_1 on \mathcal{L}_1 , and all unitary U_2 on \mathcal{L}_2 . Such superposition measures we refer to as unitarily invariant norm measures. It is straightforward to show that every unitarily invariant norm measure A_u satisfies conditions C1, C2, C3, and C4.

A. Measures from Ky-Fan norms

Given an arbitrary operator $C \in \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$ consider its singular values [22] ordered in a non-increasing sequence $s_1^{\downarrow}(C) \geq s_2^{\downarrow}(C) \geq \ldots \geq s_{N-1}^{\downarrow}(C) \geq s_N^{\downarrow}(C)$, where $N = \min[\dim(\mathcal{L}_1), \dim(\mathcal{L}_2)]$. Due to convenience we allow singular values to be zero. The Ky-Fan k-norms [24] are defined as

$$||C||_{(k)} = \sum_{l=1}^{k} s_l^{\downarrow}(C), \quad 1 \le k \le N.$$
 (66)

One may note that $||\cdot||_{(1)}$ is equal to the standard operator norm $||C||_{(1)} = \sup_{||\psi||=1} ||C|\psi\rangle||$. Moreover, $||C||_{(N)} = \text{Tr}(\sqrt{CC^{\dagger}})$ is the trace norm. In the following we write $||C||_{(\text{Tr})}$ when we wish to emphasize that we consider the special case of the trace norm.

Using the Ky-Fan norms we define the superposition measures $A_{(k)}(\rho) = ||P_1\rho P_2||_{(k)}$. We refer to these as Ky-Fan norm measures. We furthermore write $A_{(\mathrm{Tr})}$ when we want to emphasize that we consider the superposition measure from the trace norm. Since the Ky-Fan norms are unitarily invariant [24] it follows that these superposition measures forms a subclass of the unitarily invariant norm measures. However, the Ky-Fan norms have a special position among the unitarily invariant norms. Let $Q, R \in \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1)$, then $||Q||_{(k)} \leq ||R||_{(k)}$ for all k, if and only if $||Q|| \leq ||R||$ for all unitarily invariant norms $||\cdot||$ [24]. We can thus conclude the following:

Proposition 7. Let ρ and σ be arbitrary but fixed density operators on \mathcal{H} . Then $A_{(k)}(\rho) \leq A_{(k)}(\sigma)$, $\forall k$ if and only if $A_u(\rho) \leq A_u(\sigma)$, for all unitarily invariant norm measures A_u .

B. Interferometric realization of Ky-Fan norm measures

Here we consider interferometric techniques to implement all Ky-Fan norm measures. Consider a single particle that propagates in superposition between two modes, corresponding to the orthonormal elements $|1\rangle$ and $|2\rangle$. This particle also have an internal degree of freedom (e.g., polarization, spin) corresponding to the Hilbert space \mathcal{H}_I of dimension N. The total Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_s$, can be decomposed into the two orthogonal subspaces $\mathcal{H}_I \otimes \operatorname{Sp}\{|0\rangle\}$ and $\mathcal{H}_I \otimes \operatorname{Sp}\{|1\rangle\}$.

We begin with a description of an interferometric procedure to obtain the special case of the trace norm measure $A_{(\mathrm{Tr})}$. We let ρ be the total density operator on $\mathcal{H}_I \otimes \mathcal{H}_s$, and apply the unitary operator $|1\rangle\langle 1| \otimes \hat{1}_I + |2\rangle\langle 2| \otimes U$, where U is a variable unitary operator on \mathcal{H}_I . After this we apply a 50-50 beam-splitter followed by a measurement to obtain the probability that the particle is found in path 1. We find that this probability is $p = \frac{1}{2} + \mathrm{ReTr}(\langle 1|\rho|2\rangle U^{\dagger})$. Hence, by varying U until the maximal probability $p_{max} = 1/2 + ||\langle 1|\rho|2\rangle||_{(\mathrm{Tr})}$ is obtained, we can identify $A_{(\mathrm{Tr})}(\rho) = ||\langle 1|\rho|2\rangle||_{(\mathrm{Tr})} = 1/2 - p_{max}$. (Note that $A_{(\mathrm{Tr})}(\rho) \leq 1/2$, which follows from Eq. (70).) We have thus found an operational method to obtain $A_{(\mathrm{Tr})}$.

In order to obtain the other Ky-Fan norm measures we have to modify this scheme (see figure 1). In this modified scheme we first apply two independent unitary operators U and V, one in each path, $|1\rangle\langle 1|\otimes V+|2\rangle\langle 2|\otimes U$, after which we apply a 50-50 beam splitter. Consider next an arbitrary but fixed k-dimensional subspace C of \mathcal{H}_I and the corresponding projector $P_{\mathcal{C}}$. We measure the probability p_1 to find the particle in path 1 and simultaneously the internal state in subspace C, i.e., p_1 is the expectation value of $|1\rangle\langle 1|\otimes P_{\mathcal{C}}$. Alternatively, we may filter with the projector $P_{\mathcal{C}}$, after which we measure the probability to find the particle in path 1. (Note that we should not normalize after the filtering, i.e., we must keep track of the number of particles we have lost.) Similarly, we measure the probability p_2 to find the particle in path 2 and simultaneously in subspace C. This results in the probabilities $p_1 = q_1 + q_2 + r$ and $p_2 = q_1 + q_2 - r$, where

$$q_{1} = \frac{1}{2} \text{Tr}(P_{\mathcal{C}}V\langle 1|\rho|1\rangle V^{\dagger}), \ q_{2} = \frac{1}{2} \text{Tr}(P_{\mathcal{C}}U\langle 2|\rho|2\rangle U^{\dagger})$$

$$r = \text{ReTr}(P_{\mathcal{C}}V\langle 1|\rho|2\rangle U^{\dagger}), \tag{67}$$

and consequently $p_1 - p_2 = 2r = 2 \operatorname{ReTr}(P_{\mathcal{C}} V \langle 1 | \rho | 2 \rangle U^{\dagger})$. (Note that p_1 and p_2 in general do not sum to 1.) Moreover,

$$\sup_{U,V} \operatorname{ReTr}(P_{\mathcal{C}}V\langle 1|\rho|2\rangle U^{\dagger}) = ||\langle 1|\rho|2\rangle||_{(k)}, \tag{68}$$

which can be proved by making a singular value decomposition of $\langle 1|\rho|2\rangle$. Thus, if we vary the unitary operators U and V in such a way that we maximize the difference p_1-p_2 , the maximal value of this quantity is equal to

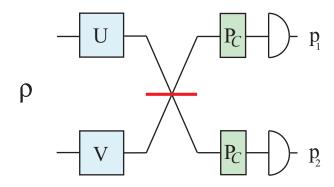


FIG. 1: (Color online) A procedure to obtain the Ky-Fan norm measure $A_{(k)} = ||\langle 1|\rho|2\rangle||_{(k)}$ by an interferometric technique. The two paths of the interferometer (upper, lower) corresponds to the orthonormal states $|1\rangle$ and $|2\rangle$, spanning the Hilbert space \mathcal{H}_s . The particle has an internal degree of freedom (e.g., polarization, spin) represented by the Hilbert space \mathcal{H}_I . The density operator ρ on the total space $\mathcal{H}_I \otimes \mathcal{H}_s$ thus describes both the spatial and internal state of the particle. First, two variable unitary operators U and V on \mathcal{H}_I are applied, one in each path, resulting in the total unitary operator $|1\rangle\langle 1|\otimes V+|2\rangle\langle 2|\otimes U$. Next, the two paths interfere at a 50-50 beam-splitter. Let $\mathcal C$ be an arbitrary but fixed k-dimensional subspace of \mathcal{H}_I , and let $P_{\mathcal{C}}$ be the corresponding projector. In both paths we filter with the projector $P_{\mathcal{C}}$, i.e., the particle is discarded if its internal state is not found within C. Next, we measure the probabilities p_1 and p_2 to find the particle in path 1 and 2, respectively. Note that due to the particle losses in the filtering, the two probabilities p_1 and p_2 do not in general sum to 1. Finally, we vary the unitary operators U and V until we find the maximal value of the difference $p_1 - p_2$, and we obtain the desired superposition measure as $A_{(k)}(\rho) = \max(p_1 - p_2)/2$.

 $2A_{(k)}(\rho)$. Hence, this interferometric technique allows us to obtain all Ky-Fan norm measures.

C. Bounds from predictability

In view of Sec. V B the Ky-Fan norm measures can be interpreted as generalized visibilities in an interferometric setup. In Refs. [25, 26] it is showed that the visibility (in the ordinary sense) is bounded by the predictability of finding the particle in the two paths. Here we show that all the Ky-Fan norm measures satisfy similar bounds. If we let \mathcal{L}_1 and \mathcal{L}_2 correspond to the two paths, and let ρ be some arbitrary state of the particle, then $p_1 = \text{Tr}(\mathcal{L}_1\rho)$ and $p_2 = \text{Tr}(\mathcal{L}_2\rho)$ are the probabilities of finding the particle in path 1 and 2, respectively, and the predictability is $\mathcal{P} = |p_1 - p_2|$.

We first note the following inequality which is a special case of Theorem IV.2.5 in [24]. If A and B are operators on the finite-dimensional Hilbert space \mathcal{H} , then

$$||AB||_{(k)} \le \sum_{l=1}^{k} s_l^{\downarrow}(A) s_l^{\downarrow}(B). \tag{69}$$

Let ρ be a density operator on \mathcal{H} and define the

"marginal" density operators $\sigma_n = \mathcal{L}_n \rho \mathcal{L}_n/p_n$, n = 1, 2. (In case $p_n = 0$, then we let σ_n be the zero operator.) Given these probabilities and marginal states we find that the Ky-Fan norm measures satisfy the bound

$$A_{(k)}(\rho) \le \sqrt{p_1 p_2} \sum_{l=1}^{k} \sqrt{\lambda_l^{\downarrow}(\sigma_1)} \sqrt{\lambda_l^{\downarrow}(\sigma_2)}, \qquad (70)$$

where $\lambda_l^{\downarrow}(\cdot)$ denotes the eigenvalues of the enclosed operator ordered nonincreasingly. Note that if the dimension of one of the subspaces is strictly larger than the other, we may add zeros to the vector with fewer eigenvalues. To prove the bound in Eq. (70) we first combine Lemma 1 with Eq. (69), to find that $||P_1\rho P_2||_{(k)} \leq \sqrt{p_1p_2} \sum_l s_l^{\downarrow}(\sqrt{\sigma_1}) s_l^{\downarrow}(D\sqrt{\sigma_2})$. Next, we use that $s_l^{\downarrow}(D\sqrt{\sigma_2}) \leq ||D||_{(1)} s_l^{\downarrow}(\sqrt{\sigma_2})$ (see, e.g., [24]) combined with $||D||_{(1)} \leq 1$. Finally, we use the fact that σ_1 and σ_2 are positive semi-definite, and thus $s_l^{\downarrow}(\sigma_n^{1/2}) = [\lambda_l^{\downarrow}(\sigma_n)]^{1/2}$, from which Eq. (70) follows.

Given the probabilities p_1, p_2 and the set of eigenvalues $\{\lambda_{1,l}^{\downarrow}\}_l$ and $\{\lambda_{2,j}^{\downarrow}\}_j$ of the two marginal operators, the bound in Eq. (70) is sharp, since one obtains equality in Eq. (70) for the density operator

$$\begin{split} \rho \; &= \; p_1 \sum_{l=1}^L \lambda_{1,l}^{\downarrow} |1:l\rangle \langle 1:l| + p_2 \sum_{j=1}^J \lambda_{2,j}^{\downarrow} |2:j\rangle \langle 2:j| \\ &+ \sqrt{p_1 p_2} \sum_{l=1}^{\min(L,J)} \sqrt{\lambda_{1,l}^{\downarrow} \lambda_{2,l}^{\downarrow}} (|1:l\rangle \langle 2:l| + |2:l\rangle \langle 1:l|), \end{split}$$

where $\{|1:l\rangle\}_{l=1}^L$ and $\{|2:j\rangle\}_{j=1}^J$ are arbitrary orthonormal bases of \mathcal{L}_1 and \mathcal{L}_2 , respectively.

Let us now take a closer look on Eq. (70). first note that $\sum_{l=1}^k \sqrt{\lambda_l^{\downarrow}(\sigma_1)} \sqrt{\lambda_l^{\downarrow}(\sigma_2)} \leq 1$, and hence $A_{(k)}(\rho) \leq \sqrt{p_1p_2} \leq 1/2$. Using this one can show that $A_{(k)}^2(\rho) + \mathcal{P}^2 \leq 1$ (although $A_{(k)}(\rho) \leq \sqrt{p_1p_2}$ gives a stronger bound on $A_{(k)}$). Hence, similarly to the ordinary visibility, the Ky-Fan norm measures are bounded by the predictability. We moreover see that the distribution of eigenvalues of the two marginal density operators put a limit on the superposition measures. For $A_{\rm (Tr)}$ the upper bound reduces to the classical fidelity between the two distributions of eigenvalues, and hence a necessary condition for $A_{(Tr)}$ to obtain the maximum $\sqrt{p_1p_2}$ (for $0 < p_1 < 1$) is that the eigenvalues of the two marginal density operators are equal. For $A_{(k)}$ to attain this maximum a necessary condition is that the marginal density operators both are of at most rank k and have the same nonzero eigenvalues. Hence, for all the Ky-Fan norms to simultaneously attain the maximum $\sqrt{p_1p_2}$, the two marginal states have to be pure. Moreover, the total state has to be pure and be possible to write as $\sqrt{p_1}|\psi_1\rangle + \sqrt{p_1}|\psi_2\rangle$, where $|\psi_1\rangle$, and $|\psi_1\rangle$ are normalized, and such that $P_1|\psi_1\rangle = |\psi_1\rangle$ and $P_2|\psi_2\rangle = |\psi_2\rangle$.

VI. SUBSPACE PRESERVING AND LOCAL SUBSPACE PRESERVING CHANNELS

With respect to a two-element decomposition $(\mathcal{L}_1, \mathcal{L}_2)$ of a Hilbert space \mathcal{H} a subspace preserving channel is such that $Tr(P_1\Phi(\rho)) = Tr(P_1\rho)$ for all density operators ρ on \mathcal{H} [11]. If the decomposition corresponds to the two paths of a Mach-Zehnder interferometer, as described in Sec. VB, this condition means that there is no transfer of the particle from one path to the other; the probability weights on the two paths are preserved under the action of this class of channels. Apart from this restriction we can use any type of information transfer or sharing of correlated resources between the two paths when we generate these channels. The local subspace preserving channels [11] forms the subset of the SP channels that can be generated using only local operations at the two paths of the interferometer. Another way to put this is to say that the LSP channels are those SP channels that can be obtained from product channels on the second quantization of \mathcal{H} . In terms of the mapping M introduced in Sec. IV an SP channel Φ is LSP if there exist a product channel $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ on $\widetilde{\mathcal{H}}$ such that $\Phi(\rho) = M^{\dagger} [\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2] (M \rho M^{\dagger}) M$. A more detailed introduction to these concepts can be found in Ref. [11], and their use in the context of interferometry can be found in Refs. [13, 14]. Here we consider the relation between superposition measures and these two classes of channels.

A. Induced measures

Proposition 8. Let \mathcal{L} be a decomposition of \mathcal{H} , and suppose Φ is a channel on \mathcal{H} . Then

$$A_S^{\mathcal{L}}(\Phi(\rho)) \le A_S^{\mathcal{L}}(\rho),$$
 (71)

for all density operators ρ on \mathcal{H} , if and only if

$$\Pi_{\mathcal{L}} \circ \Phi \circ \Pi_{\mathcal{L}} = \Phi \circ \Pi_{\mathcal{L}}. \tag{72}$$

Here, "o" denotes compositions of mappings. In words this proposition states that a channel does not increase the relative entropy of superposition (for any input state) if and only if it maps all block diagonal states to block diagonal states. One may note a similar relation between relative entropy of entanglement and separable channels [27, 28, 29, 30].

Proof. For the "only if" part of the proof we note that $\Pi(\rho)$ is block diagonal, and thus (since A_S satisfies C2) $A_S(\Pi(\rho)) = 0$. From Eq. (71), it follows that $A_S(\Phi \circ \Pi(\rho)) = 0$. Thus, according to condition C2, it follows that $\Phi \circ \Pi(\rho)$ is block-diagonal, i.e., Eq. (72) follows.

For the "if" part of the proof we note the trivially satisfied relation $S(\Pi \circ \Phi(\rho)||\Pi \circ \Phi \circ \Pi(\rho)) \geq 0$. For any density operators w and σ it holds that $\text{Tr}(w \ln \Pi(\sigma)) = \text{Tr}(\Pi(w) \ln \Pi(\sigma))$. Hence, we can rewrite the previous inequality as $\text{Tr}(\Phi(\rho) \ln \Pi \circ \Phi(\rho)) \geq \text{Tr}(\Phi(\rho) \ln \Pi \circ \Phi \circ \Pi(\rho))$. This we combine with Eq. (72) to obtain $\text{Tr}(\Phi(\rho) \ln \Pi \circ \Phi(\rho)) = \text{Tr}(\Phi(\rho) \ln \Pi \circ \Phi(\rho))$.

 $\Phi(\rho)$) $\geq \operatorname{Tr}(\Phi(\rho) \ln \Phi \circ \Pi(\rho))$, which can be used to show that $A_S(\Phi(\rho)) \leq S(\Phi(\rho)||\Phi \circ \Pi(\rho))$. Now we use the contractivity of relative entropy [21], which implies that $S(\Phi(\rho)||\Phi \circ \Pi(\rho)) \leq S(\rho||\Pi(\rho)) = A_S(\rho)$. Thus follows Eq. (71) and the proposition is proved. \square

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ be a decomposition of \mathcal{H} .

- For a channel Φ that is SP with respect to \mathcal{L} it holds that $A_S^{\mathcal{L}}(\Phi(\rho)) \leq A_S^{\mathcal{L}}(\rho)$, for every density operator ρ on \mathcal{H} .
- For every superposition measure A that is induced by a bipartite entanglement measure it holds that $A^{\mathcal{L}}(\Phi(\rho)) \leq A^{\mathcal{L}}(\rho)$, for every density operator ρ on \mathcal{H} , and every channel Φ that is LSP with respect to \mathcal{L} .

To show that SP channels do not increase the relative entropy of superposition we note that if a channel is SP it follows, [11] Proposition 4, that it can be written

$$\Phi(\rho) = P_1 \Phi(P_1 \rho P_1) P_1 + P_2 \Phi(P_2 \rho P_2) P_2
+ P_1 \Phi(P_1 \rho P_2) P_2 + P_2 \Phi(P_2 \rho P_1) P_1$$
(73)

This can be used to show that Φ satisfies Eq. (72). Hence, according to Proposition 8 it follows that Φ cannot increase the relative entropy of superposition. To show that LSP channels do not increase any induced superposition measure we note that since LSP channels correspond to special types of product channels on the second quantized spaces [11]. It directly follows that the superposition measures induced by entanglement measures cannot increase under LSP channels, if we make the natural assumption that entanglement measures do not increase under local operations.

B. Norm measures

Lemma 7. Let $\{V_k\}_{k=1}^K$ and $\{W_l\}_{l=1}^L$ be operators on \mathcal{H} be such that $\sum_k V_k^{\dagger} V_k \leq \hat{1}$ and $\sum_l W_l^{\dagger} W_l \leq \hat{1}$, and let C be a complex $K \times L$ matrix such that $CC^{\dagger} \leq I$. Then,

$$||\sum_{kl} C_{kl} V_k Q W_l^{\dagger}||_{(\text{Tr})} \le ||Q||_{(\text{Tr})},$$
 (74)

for all linear operators Q on \mathcal{H} .

In other words this lemma states that the trace norm of an operator cannot increase under this type of transformations.

Proof. We begin with a singular value decomposition $C = U^{(a)}SU^{(b)\dagger}$, where $S_{kl} = s_k^{\downarrow}(C)\delta_{kl}$ for $k, l \leq \min(K, L)$ and $S_{kl} = 0$ otherwise. We use the unitary matrices $U^{(a)}$ and $U^{(b)}$ to define $\overline{V}_{k'} = \sum_k V_k U_{kk'}^{(a)}$ and $\overline{W}_{l'} = \sum_l W_l U_{ll'}^{(b)}$. It follows that

$$\sum_{kl} C_{kl} V_k Q W_l^{\dagger} = \sum_{n=1}^{\min(K,L)} s_n^{\downarrow}(\mathbf{C}) \overline{V}_n Q \overline{W}_n^{\dagger}, \qquad (75)$$

for all linear operators Q on \mathcal{H} . We furthermore find that $\sum_n \overline{V}_n^{\dagger} \overline{V}_n \leq \hat{1}$ and $\sum_n \overline{W}_n^{\dagger} \overline{W}_n \leq \hat{1}$. One can show that

$$||\sum_{n} s_{n}^{\downarrow}(\boldsymbol{C}) \overline{V}_{n} Q \overline{W}_{n}^{\dagger}||_{(\mathrm{Tr})} \leq \sum_{n} ||\overline{V}_{n} Q \overline{W}_{n}^{\dagger}||_{(\mathrm{Tr})}, \quad (76)$$

where we have used that the condition $CC^{\dagger} \leq I$ implies $s_n^{\downarrow}(C) \leq 1$. Consider now a singular value decomposition $Q = \sum_j s_j^{\downarrow}(Q)|f_j\rangle\langle d_j|$, where $\{|f_j\rangle\}_j$ and $\{|d_j\rangle\}_j$ both are orthonormal sets.

$$||\overline{V}_{n}Q\overline{W}_{n}^{\dagger}||_{(\mathrm{Tr})} \leq \sum_{j} s_{j}^{\downarrow}(Q)||\overline{V}_{n}|f_{j}\rangle\langle d_{j}|\overline{W}_{n}^{\dagger}||_{(\mathrm{Tr})}$$

$$= \sum_{j} s_{j}^{\downarrow}(Q)\sqrt{\langle f_{j}|\overline{V}_{n}^{\dagger}\overline{V}_{n}|f_{j}\rangle}$$

$$\times \sqrt{\langle d_{j}|\overline{W}_{n}^{\dagger}\overline{W}_{n}|d_{j}\rangle}$$

$$(77)$$

Now we use the Cauchy-Schwartz inequality to find that $\sum_n \langle f_j | \overline{V}_n^\dagger \overline{V}_n | f_j \rangle^{1/2} \langle d_j | \overline{W}_n^\dagger \overline{W}_n | d_j \rangle^{1/2} \leq 1$. If this is combined with Eqs. (75) - (77), and with the fact that $||Q||_{(\mathrm{Tr})} = \sum_j s_j^\dagger(Q)$, we find Eq. (74) and have thus proved the lemma. \square

Proposition 9. Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ be a two-element decomposition of \mathcal{H} , and suppose Φ is a channel on \mathcal{H} . Then

$$A_{(\mathrm{Tr})}^{\mathcal{L}}(\Phi(\rho)) \le A_{(\mathrm{Tr})}^{\mathcal{L}}(\rho),$$
 (78)

for all density operators ρ on \mathcal{H} , if and only if

$$\Pi_{\mathcal{L}} \circ \Phi \circ \Pi_{\mathcal{L}} = \Phi \circ \Pi_{\mathcal{L}}. \tag{79}$$

Note that $A_{(Tr)}$ denotes the superposition measure obtained from the trace norm $||\cdot||_{(Tr)}$.

Proof. For the "only if" part of the proof we note that $\Pi(\rho)$ is block diagonal, and thus $A_{(\text{Tr})}(\Pi(\rho)) = 0$, since $A_{(\text{Tr})}$ satisfies C2. From Eq. (78), it follows that $A_{(\text{Tr})}(\Phi \circ \Pi(\rho)) = 0$. Thus, according to condition C2, it follows that $\Phi \circ \Pi(\rho)$ is block-diagonal, and Eq. (79) follows.

For the "if" part of the proof we define the linear map $\Pi^{\perp}(Q) = P_1QP_2 + P_2QP_1$, for all linear operators Q on \mathcal{H} . (Note that Π^{\perp} is not a CPM, and not even a positive map.) One can show that the condition in Eq. (79) is equivalent to $\Pi^{\perp} \circ \Phi \circ \Pi^{\perp} = \Pi^{\perp} \circ \Phi$. Now we note that Π^{\perp} can be written such that it satisfies the conditions in Lemma 7 (with C the identity matrix and $V_1 = P_1$, $V_2 = P_2$, $W_1 = P_2$, and $W_2 = P_1$). Moreover, one can note that Φ (like every other channel) also satisfies the conditions in Lemma 7. Thus, we let ρ be an arbitrary density operator on \mathcal{H} , and we use Lemma 7 twice to obtain

$$||\Pi^{\perp} \circ \Phi(\rho)||_{(\operatorname{Tr})} = ||\Pi^{\perp} \circ \Phi \circ \Pi^{\perp}(\rho)||_{(\operatorname{Tr})}$$

$$\leq ||\Phi \circ \Pi^{\perp}(\rho)||_{(\operatorname{Tr})}$$

$$\leq ||\Pi^{\perp}(\rho)||_{(\operatorname{Tr})}. \tag{80}$$

Now we note that for each Hermitian operator R it holds that $||\Pi^{\perp}(R)||_{(\mathrm{Tr})} = 2||P_1RP_2||_{(\mathrm{Tr})}$. If we combine this with Eq. (80) we find that $2A_{(\mathrm{Tr})}(\Phi(\rho)) = ||\Pi^{\perp} \circ \Phi(\rho)||_{(\mathrm{Tr})} \leq ||\Pi^{\perp}(\rho)||_{(\mathrm{Tr})} = 2A_{(\mathrm{Tr})}(\rho)$, which proves the proposition. \square

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ be a decomposition of \mathcal{H} .

- For a channel Φ that is SP with respect to \mathcal{L} it holds that $A_{(\mathrm{Tr})}^{\mathcal{L}}(\Phi(\rho)) \leq A_{(\mathrm{Tr})}^{\mathcal{L}}(\rho)$, for every density operator ρ on \mathcal{H} .
- For every unitarily invariant norm measure A_u it holds that $A_u^{\mathcal{L}}(\Phi(\rho)) \leq A_u^{\mathcal{L}}(\rho)$, for every density operator ρ on \mathcal{H} , and every channel Φ that is LSP with respect to \mathcal{L} .

Concerning the proofs of these statements note that we have already shown in Sec. VIA that all SP channels satisfy the condition in Eq. 79, and thus SP channels cannot increase $A_{(Tr)}$. It remains to prove that LSP channels do not increase any unitarily invariant norm measure. From Proposition 30 in Ref. [11] we know that if Φ is LSP with respect to \mathcal{L} then it follows that $P_1\Phi(\rho)P_2 = V\rho W^{\dagger}$, where $V = \sum_k c_{1,k}V_k$ and W = $\sum_{l} c_{l} W_{l}, \text{ and where } \sum_{k} V_{k}^{\dagger} V_{k} = P_{1}, \sum_{k} V_{k}^{\dagger} V_{k} = P_{2},$ $\sum_{k} |c_{1,k}|^{2} \leq 1, \text{ and } \sum_{l} |c_{2,l}|^{2} \leq 1. \text{ Moreover, } P_{1} V_{k} P_{1} =$ $V_{k} \text{ and } P_{2} W_{k} P_{2} = W_{l}. \text{ From these conditions it follows}$ that $||V||_{(1)} \leq 1$ and $||W||_{(1)} \leq 1$. By using Proposition IV.2.4 in Ref. [24] we find that $||VP_1\rho P_2W^{\dagger}||_{(k)} \leq$ $||V||_{(1)}||P_1\rho P_2||_{(k)}||W^{\dagger}||_{(1)} \le ||P_1\rho P_2||_{(k)}$. Thus, we can conclude that $A_{(k)}(\Phi(\rho)) \leq A_{(k)}(\rho)$. Since this holds for all k, it follows, according to Proposition 7, that $A_u(\Phi(\rho)) \leq A_u(\rho)$ for all unitarily invariant norm measures.

In view of the results in this section there appears to be a correspondence between the class of induced measures and the class of invariant norm measures, and especially between A_S and $A_{(Tr)}$, as is evident from Propositions 8 and 9. However, it is far from clear to what extent this indicates a profound relation between these measures. We finally note that Propositions 8 and 9 shows that SP channels are limited in their capacity to create superposition since they cannot increase A_S and A_{Tr} . However, as seen in Sec. VII, the other Ky-Fan norm measures may increase under the action of SP channels.

VII. ATOM UNDERGOING RELAXATION IN AN INTERFEROMETER

To illustrate the theory we consider models of an atom undergoing relaxation while propagating in a Mach-Zehnder interferometer. We apply the Ky-Fan norm measures to follow the evolution of the degree of superposition between the two paths. We assume that the atom has N relevant energy eigenstates spanning \mathcal{H}_I , e.g., electronic states, or nuclear spin states in an external magnetic field.

To model the relaxation we let the total density operator ρ , describing both the internal and spatial degree of freedom, evolve according to a time-independent master equation that can be written on the Lindblad form [31],

$$\frac{d}{dt}\rho = F(\rho),$$

$$F(\rho) = -i[H \otimes \hat{1}_s, \rho] + \sum_k L_k \rho L_k^{\dagger}$$

$$-\frac{1}{2} \sum_k L_k^{\dagger} L_k \rho - \frac{1}{2} \sum_k \rho L_k^{\dagger} L_k, \qquad (81)$$

where H is the Hamiltonian of the atom, $\hat{1}_s$ is the identity operator on \mathcal{H}_s , and L_k are the Lindblad operators. For convenience we assume $\hbar = 1$ and that energies are measured in units of some reference energy E, and likewise the dimensionless parameter t measures time in units of E^{-1} . The Lindblad form guarantees that the evolution $\rho(t_1) = \Phi_{t_1-t_0}(\rho(t_0))$, for $t_1 \geq t_0$, is such that the one-parameter family of dynamical maps Φ_s are channels [31].

We first note that there is a sufficient condition for the dynamical maps Φ_s of Eq. (81) to be SP with respect to the decomposition $(\mathcal{L}_1, \mathcal{L}_2)$. One can see that if $\text{Tr}[P_1F(\rho)] = 0$ for all density operators ρ , then $\frac{d}{dt}\text{Tr}(P_1\rho) = 0$, which implies that all dynamical maps Φ_s are SP. A sufficient condition for $\text{Tr}[P_1F(\rho)] = 0$ in Eq. (81) is that $\Pi(L_k) = L_k$.

We first consider two examples that admit analytical solutions. Let $H = \hat{0}$ and assume the Lindblad operators $L_k = \sqrt{g}|e_1\rangle\langle e_k|\otimes \hat{1}_s$, for $k=1,2,\ldots,N$, where $\{|e_k\rangle\}_{k=1}^N$ is an orthonormal eigenbasis of \mathcal{H}_I . Hence, all states relax to the ground state (a slight misnomer since $H = \hat{0}$ at the same rate. The internal degree of freedom of the atom is prepared in the maximally mixed state (e.g., by letting it equilibrate at a sufficiently high temperature), and apply a 50-50 beamsplitter to obtain the state $\rho(0) = \hat{1}_N \otimes |\psi\rangle\langle\psi|/N$, where $|\psi\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. Given this initial state we let the system evolve according to the master equation in Eq. (81), and thus propagate the atom at a sufficiently low temperature to allow relaxation to the ground state. (Again, this description is not entirely appropriate since $H = \hat{0}$, but is more adequate when we later let H be nonzero.) By solving the master equation one finds that $A_{(1)}(\rho(t)) = 1/(2N) + (1 - e^{-gt})(N - 1)/(2N),$ $A_{(k)}(\rho(t)) = 1/2 - e^{-gt}[1/2 - k/(2N)], \text{ for } k = 2, \dots, N.$ Especially, we find that $A_{(Tr)}(\rho(t)) \equiv A_{(N)}(\rho(t)) = 1/2$. Hence, $A_{(Tr)}$ is constant, which is consistent with the results in Sec. VIB. It is to be noted that SP channels in general require nonlocal resources for their implementation, thus the increasing superposition as measured by the other Ky-Fan norm measures should not come as a surprise. As seen, all the Ky-Fan norm measures approach their maximal value 1/2, which we know from Sec. V C can be attained only if the total state is pure. This is indeed the case since the internal state approaches the ground state, and does so in a manner that does not disrupt the superposition between the two paths.

For the second example we still assume $H = \hat{0}$, but let $L_{1:k} = \sqrt{g}|e_1\rangle\langle e_k|\otimes |1\rangle\langle 1|$, and $L_{2:k} = \sqrt{g}|e_1\rangle\langle e_k|\otimes |2\rangle\langle 2|$, for $k=1,2,\ldots,N$, constitute the set of Lindblad operators. In this case the superposition measures decay exponentially as $A_{(k)}(\rho(t)) = ke^{-gt}/(2N)$, $k=1,2,\ldots,N$.

To a obtain a slightly more realistic model we assume a nonzero Hamiltonian and arbitrary decay rates between all eigenstates. We consider the superoperator F_1 as in Eq. (81) with the Lindblad operators

$$L_{kk'} = \sqrt{g_{kk'}} |e_k\rangle\langle e_{k'}| \otimes \hat{1}_s, \quad 1 \le k \le k' \le N, \quad (82)$$

where $|e_k\rangle$ are the eigenstates of the Hamiltonian ordered in increasing energy. Figure 2 shows the result of a numerical calculation, where the solid curves correspond to the evolution of the Ky-Fan norm measures $A_{(3)}$, $A_{(2)}$, and $A_{(1)}$, counted from the top and downward. (For the sake of clarity we consider a three-level system.) The eigenvalues of the Hamiltonian H and the coefficients $g_{kk'}$ have been selected randomly. We next consider the master equation with superoperator F_2 , obtained from the Lindblad operators

$$L_{1:kk'} = \sqrt{g_{kk'}} |e_k\rangle\langle e_{k'}| \otimes |1\rangle\langle 1|,$$

$$L_{2:kk'} = \sqrt{g_{kk'}} |e_k\rangle\langle e_{k'}| \otimes |2\rangle\langle 2|,$$
(83)

for $1 \leq k \leq k' \leq N$. We use the same Hamiltonian, coefficients $g_{kk'}$, and initial state as in the previous example. In FIG. 2 the evolution of the superposition measures correspond to the dashed curves. Finally, we consider the master equation with the superoperator $F_3 = 0.8F_1 + 0.2F_2$, i.e., a convex combination of the two previous master equations. We use the same initial state as before, and obtain the evolution of the three Ky-Fan norm measures as depicted in FIG. (2) with dotted lines.

As seen by the examples, there are two extreme cases: the "nonlocal" relaxation where all the Ky-Fan norm measures approach the maximal value 1/2, and the "local" relaxation where they approach the minimal value 0. One can show that the master equation of the local relaxation F_2 gives dynamical mappings that are LSP, while the nonlocal relaxation gives dynamical mappings that are general SP channels. For master equations there is an implicitly assumed environment causing the nonunitary evolution. In the nonlocal case the master equation does not distinguish the path states, which we may interpret as the environment being insensitive to the path state of the atom. That there is no path information stored in the environment seems consistent with the nonvanishing superposition. In the case of local relaxation, however, local environments distinguish the paths and should reasonably cause decay of the superposition, as we indeed see is the case in FIG. 2. In the third case, with F_3 , some of the dotted curves in FIG. (2) do increase beyond their initial value. It follows (by the results in Sec. VIB) that the intermediate case (corresponding to F_3) cannot be LSP and thus represents a nonlocal relaxation. To summarize these examples, it appears as if the Ky-Fan norm measures to some extent reflect the locality or nonlocality of the relaxation process.

One might wonder to what extent the nonlocal decay model is realistic. If the internal degree of freedom is a nuclear spin in an external magnetic field, and we assume that the system relaxates via spontaneous emission, the small energy splitting results in a long wavelength of the emitted photons. Moreover, these states have a very long life time in general, which suggests a significant coherence length of the emitted radiation. If both these length scales are much larger than the separation of the two paths one might speculate that the decay process does not "notice" the path difference, and thus be nonlocal in the above sense. (There may, however, be other decoherence mechanisms that cause localization.) Along these lines one could consider experiments in order to test to what degree, and on what length scales, different relaxation mechanisms of various excited internal degrees of freedom do localize an atom.

In the above discussions we have interpreted the two paths of the interferometer as two spatial modes. However, we could also consider time binning (see, e.g., Ref. [32] and a discussion in Ref. [14]), two interfering decay channels, or some other degree of freedom. It is only required that there is no significant transition between the orthogonal subspaces corresponding to the two "paths".

VIII. CONCLUSIONS

We introduce the concept of superposition measures with respect to given orthogonal decompositions of the Hilbert space of a quantum system, which can be regarded as an analogue to entanglement measures with respect to decompositions into subsystems. By a second quantization of the system, superposition can be regarded as entanglement, and thus makes it possible to construct superposition measures using entanglement measures. We find the superposition measures induced by relative entropy of entanglement and of entanglement of formation, and obtain a decomposition formula of the relative entropy of entanglement for certain classes of states. We furthermore consider a class of superposition measures based on unitarily invariant operator norms. Especially we consider measures derived from Ky-Fan norms, and show that these can be obtained operationally using interferometric techniques. We show that the Ky-Fan norm based measures are bounded by predictability, similarly as for the standard visibility in interferometry [25, 26]. We furthermore consider the superposition measures under the action of subspace preserving and local subspace preserving channels [11], and show that these channels cannot increase the superposition with respect to certain superposition measures. We illustrate the theory with models of an atom undergoing relaxation while propagating in a Mach-Zehnder interferometer. We consider "local" and "nonlocal" relaxation models, and monitor this difference by the evolution of the superposition measures.

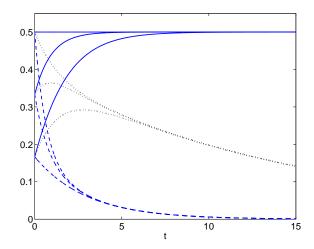


FIG. 2: (Color online) Evolution of the Ky-Fan norm measures of the solution of a master equation describing an atom undergoing relaxation in the two paths of a Mach-Zehnder interferometer. The superposition measures are calculated with respect to the two paths. Counted from the top and down, the solid curves depict $A_{(3)}(\rho(t)) \ge A_{(2)}(\rho(t)) \ge A_{(1)}(\rho(t))$, where $\rho(t)$ is the solution of the master equation in Eq. (81) where the superoperator F_1 (the "nonlocal" relaxation) is defined with respect to the Lindblad operators in Eq. (82). For convenience, the master equation is formulated such that t is a dimensionless parameter. The three dashed curves depict the evolution of the same superposition measures, but for a master equation with the superoperator F_2 (the "local" relaxation) determined by the Lindblad operators in Eq. (83). The three dotted lines gives the evolution resulting from the superoperator $F_3 = 0.8F_1 + 0.2F_2$, i.e., a convex combination of the two previous master equations. In all cases the internal input state of the atom is maximally mixed. The atom is put in superposition between the two paths using a 50-50beam-splitter. The Hamiltonian and the coefficients $g_{kk'}$ in Eqs. (82) and (83) are chosen randomly, but are the same for all three cases. As seen, the local relaxation model appears to remove all superposition, while for the nonlocal relaxation all the Ky-Fan norm measures appear to approach their maximal value 1/2. For the intermediate case F_3 , two of the Ky-Fan norm measures initially increase, thus revealing the nonlocal nature of this case, since the corresponding channels cannot be LSP.

One could consider to extend the ideas presented here

by defining a "superposition cost" by inducing it from the entanglement cost [23]. Similarly, one can induce a "distillable superposition" from distillable entanglement [10]. It would be interesting to find formulations of these induced superposition measures directly in terms of the subspace decompositions of \mathcal{H} , similarly as done here for the relative entropy of superposition and superposition of formation, rather than via the indirect definition using the second quantization.

The quantitative approach to superposition introduced here may facilitate the development of interferometry as a probing technique of processes in physical systems [11, 12, 13, 14, 15]. Especially, one could consider to use the superposition measures to systematically construct interferometric channel measures in the spirit of Refs. [12, 13, 14].

One could furthermore consider to combine the superposition measure concept introduced here, and the channel measures in Refs. [12, 13, 14], with the perspective put forward in Ref. [16]. The interference measure introduced in Ref. [16] quantifies how sensitive the outcomes of measurements in the computational basis are to phase changes of input superpositions to a quantum process. It does not seem unreasonable that there exist relations between this interference measure and the change of superposition caused by the channel. Such a comparison would benefit from a generalization of the measure put forward in Ref. [16] to the general type of subspace decompositions considered here.

Superposition measures may also be useful to the efforts to find meaningful generalizations of geometric phases and holonomies [33, 34] to mixed states and open systems [35, 36, 37, 38, 39, 40, 41, 42, 43]. One may for example note that in Ref. [20] the Uhlmann holonomy [44] have been formulated within a framework closely related to the type of subspace structures considered here.

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